Notes on Random Processes

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1 Poisson Process

Definition 1 (Counting Process)**.** A counting process $\{N_t\}_{t\geq 0}$ is a collection of non-negative, integer-valued random variables such that if $0 \leq s \leq t$, then $N_s \leq N_t$.

Definition 2 (Poisson Process)**.** A Poisson process with parameter λ is a counting process $\{N_t\}_{t\geq 0}$ with the following properties:

1. Stationarity : $N_t \overset{D}{=} N_{s+t} - N_s$ with $N_0 = 0$.
2. Independent Increment : $N_t,\ N_{t+h} - N_t$ are independent.
3. Proportionality : $\mathbb{P}(N_h = 1) = \lambda h + o(h)$ as $h \to 0$.
4. Rareness : $\mathbb{P}(N_h \geq 2) = o(h)$ as $h \to 0$.

Theorem 1 (Distribution of N_t)**.** *Let $\{N_t\}_{t\geq 0}$ be a Poisson process. Then*

$$N_t \sim \mathit{Poisson}(\lambda t)$$

Proof. Since $\{N_t\}_{t\geq 0}$ is a counting process, we have

$$\begin{aligned}
\mathbb{P}(N_{t+h} = x) &= \sum_{k=0}^{x} \mathbb{P}(N_{t+h} = x, N_t = k) \\
&= \mathbb{P}(N_{t+h} = x, N_t = x) + \mathbb{P}(N_{t+h} = x, N_t = x-1) + \mathbb{P}(N_{t+h} = x, N_t \leq x-2) \\
&= \mathbb{P}(N_{t+h} - N_t = 0, N_t = x) + \mathbb{P}(N_{t+h} - N_t = 1, N_t = x-1) + \mathbb{P}(N_{t+h} - N_t \geq 2, N_{t+h} = x) \\
&= \mathbb{P}(N_h = 0)\mathbb{P}(N_t = x) + \mathbb{P}(N_h = 1)\mathbb{P}(N_t = x-1) + o(h) \\
&= \mathbb{P}(N_t = x)\{1 - (\lambda h + o(h)) - o(h)\} + \mathbb{P}(N_t = x-1)\{\lambda h + o(h)\} + o(h),
\end{aligned}$$

where the fourth equality follows from the stationarity and independence of increments, together with the rareness condition of the Poisson process. And the fifth equality follows since

$$\mathbb{P}(N_h = 0) = 1 - \mathbb{P}(N_h = 1) - \mathbb{P}(N_h \geq 2)$$

Let $g(x,t) := \mathbb{P}(N_t = x)$, then

$$g(x, t+h) = g(x,t)\{1 - (\lambda h + o(h)) - o(h)\} + g(x-1,t)\{\lambda h + o(h)\} + o(h).$$

Thus, we have

$$\begin{aligned}
\frac{d}{dt} g(x,t) &= \lim_{h\to 0} \frac{g(x,t+h) - g(x,t)}{h} \\
&= -\lambda g(x,t) + \lambda g(x-1,t).
\end{aligned}$$

Now, by famous fact from first order ODE, we can observe that

$$\begin{aligned} & e^{\lambda t}\frac{d}{dt}g(x,t) + \lambda e^{\lambda t}g(x,t) = \lambda e^{\lambda t}g(x-1,t) \\ & \Rightarrow \frac{d}{dt}\{e^{\lambda t}g(x,t)/\lambda^x\} = \lambda e^{\lambda t}g(x-1,t)/\lambda^x. \end{aligned}$$

Finally, by the fact that $N_0 = 0$, we can observe $g(0,0) = 1, g(1,0) = g(2,0) = \cdots = 0$.

$$\begin{aligned} & g'(0,t) = -\lambda g(0,t) + \lambda g(-1,t) = -\lambda g(0,t) \\ & \Rightarrow g(0,t) = e^{-\lambda t} \\ & \Rightarrow \frac{d}{dt}\{e^{\lambda t}g(1,t)/\lambda^1\} = \lambda e^{\lambda t}g(0,t)/\lambda^1 = 1 \\ & \Rightarrow g(1,t) = \lambda t e^{-\lambda t} \end{aligned}$$

Recursively, we can get

$$\mathbb{P}(N_t = x) = g(x,t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}, \quad x = 0, 1, \cdots$$

□

Theorem 2 (Distribution of Waiting-Time)**.** *Let W_r denote the waiting time until the r-th arrival in the Poisson process $\{N_t\}_{t\geq 0}$ with rate λ. Then*

$$W_r \sim \mathit{Gamma}\left(r, \frac{1}{\lambda}\right).$$

Furthermore, the interarrival times

$$W_1,\ W_2 - W_1,\ \ldots,\ W_r - W_{r-1}$$

are i.i.d. and satisfy

$$W_1,\ W_2 - W_1,\ \ldots,\ W_r - W_{r-1} \overset{\text{iid}}{\sim} \mathit{Exp}\left(\frac{1}{\lambda}\right).$$

Proof. Let $X_j := W_j - W_{j-1}$ for $j \geq 1$, with $W_0 := 0$, so that in particular $X_1 = W_1$.
First, $X_1 \sim \mathsf{Exp}\left(\frac{1}{\lambda}\right)$, since

$$\{X_1 > t\} = \{W_1 > t\} = \{N_t = 0\},$$

and hence

$$\mathbb{P}(X_1 > t) = \mathbb{P}(N_t = 0) = e^{-\lambda t}.$$

Next, for $s, t \geq 0$, conditioning on $X_1 = t$, we have

$$\{X_2 > s\} = \{W_2 - W_1 > s\} = \{N_{t+s} - N_t = 0\}.$$

By the stationary and independent increments property of the Poisson process,

$$N_{t+s} - N_t \sim \mathsf{Poisson}(\lambda s), \qquad N_{t+s} - N_t \perp N_t.$$

Therefore,

$$\mathbb{P}(X_2 > s \mid X_1 = t) = \mathbb{P}(N_{t+s} - N_t = 0) = e^{-\lambda s}.$$

Since this does not depend on t, it follows that

$$X_2 \sim \mathsf{Exp}\left(\frac{1}{\lambda}\right), \qquad X_2 \perp X_1.$$

More generally, for $n \geq 1$, conditioning on $W_n = t_n$, we have

$$\{X_{n+1} > s\} = \{W_{n+1} - W_n > s\} = \{N_{t_n+s} - N_{t_n} = 0\}.$$

Hence,

$$\mathbb{P}(X_{n+1} > s \mid W_n = t_n) = \mathbb{P}(N_{t_n+s} - N_{t_n} = 0) = e^{-\lambda s}.$$

Since W_n is a function of $X_1, \ldots, X_n$, this implies

$$\mathbb{P}(X_{n+1} > s \mid X_1, \ldots, X_n) = e^{-\lambda s}.$$

Therefore,

$$X_{n+1} \sim \mathsf{Exp}\left(\frac{1}{\lambda}\right), \qquad X_{n+1} \perp (X_1, \ldots, X_n).$$

By induction, $X_1, X_2, \ldots$ are i.i.d. $\mathsf{Exp}\left(\frac{1}{\lambda}\right)$.
Finally,

$$W_r = \sum_{j=1}^{r} X_j,$$

and since the sum of r i.i.d. $\mathsf{Exp}\left(\frac{1}{\lambda}\right)$ random variables is $\mathsf{Gamma}\left(r, \frac{1}{\lambda}\right)$, we conclude that

$$W_r \sim \mathsf{Gamma}\left(r, \frac{1}{\lambda}\right).$$

□

1.1 Uniform Distribution

Our other interest is in the distribution of arrival times of a Poisson process S_n when $N_t = n$ is observed.

Theorem 3 (Distribution of S_T)**.** *Let $S_1, S_2, \ldots, S_n$ be the arrival times of a Poisson process with parameter λ. Then*

$$f_{S_1, S_2 \ldots, S_n | N_t = n}(s_1, s_2, \ldots, s_n) = \frac{n!}{t^n} \mathbb{I}\{0 < s_1 < \cdots < s_n < t\},$$

which implies $(S_1, \ldots, S_n) \overset{D}{=} (U_{(1)}, \ldots, U_{(n)})$.

Proof. See Page 269 of this. □

1.2 Bernoulli Process

Definition 3 (Bernoulli Process)**.** A Bernoulli process with parameter $\theta \in [0, 1]$ is a (discrete-time) counting process $\{S_n\}_{n \geq 0}$ defined by

$$S_0 = 0, \qquad S_n = \sum_{j=1}^{n} Z_j \quad (n \geq 1),$$

where

$$Z_1, Z_2, \ldots \overset{\text{iid}}{\sim} \mathsf{Bernoulli}(\theta).$$